ON A CONJECTURE REGARDING NONSTANDARD
UNISERIAL MODULES

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ABSTRACT. We consider the question of which valuation domains (of cardinality $\aleph_1$) have nonstandard uniserial modules. We show that a criterion conjectured by Osofsky is independent of $\text{ZFC} + \text{GCH}$.

INTRODUCTION

The story of the study of nonstandard uniserial modules is a long and interesting one, which we outline only briefly here; the reader may consult [BS1], [FS], or [O2] for more information. The existence of nonstandard uniserial modules over some valuation domains was first proved by the second author [Sh], by forcing a model of ZFC with a nonstandard uniserial module, and then using a completeness theorem for stationary logic to show that such examples exist in all models of ZFC, i.e., the existence is a consequence of ZFC. Then Fuchs and Salce [FS] constructed nonstandard uniserial modules using the diamond principle, which is consistent with ZFC. Fuchs noticed that a nonstandard divisible uniserial module could be used to construct an affirmative answer to Kaplansky's question whether there is a valuation ring which is not the quotient of a valuation domain. Fuchs and Shelah [FSh] used the compactness theorem for the logic with quantification over branches to give another proof that divisible nonstandard uniserials exist in all models of ZFC. The first author [E] elaborated on this argument to give a general transfer principle that, for example, showed that various different classes of nonstandard uniserials constructed by Bazzoni and Salce [BS1, 2, 3] using $\diamondsuit$ existed in all models of ZFC.

Now this use of theorems from mathematical logic did not bother the second author; he was happy if algebraists had to learn logic! However, in this view he was probably a singleton among those interested in the problem. Aside from eliminating the methods of logic from the proof, the hope was that a more explicit proof would give more information, e.g. an understanding of exactly

which valuation domains had nonstandard uniserial modules. Osofsky [O1, 2] was the first to give a concrete construction (in ZFC) of valuation domains with nonstandard uniserials. She ventured a conjecture [O1] as to which valuation domains had nonstandard uniserials; we will state it below in a modified form. It is a principal result of this paper that, while this conjecture is correct assuming $V = L$ (the Axiom of Constructibility), it fails in some other models of $ZFC + GCH$. Thus, it seems, logic cannot be expelled from the subject!

It is well known that if R is "complete" in some sense (e.g., almost maximal), then there are no nonstandard uniserial R-modules. Here we introduce a couple of invariants ("Gamma invariants") of a ring whose values are stationary subsets of ω_1—or, more precisely, equivalence classes of stationary subsets under the equivalence relation of equality on a closed unbounded subset. These invariants can be viewed as measuring (the lack of) "completeness" of the ring. Some results about implications between the values of these invariants and the existence of nonstandard uniserials are proved in ZFC, but other implications are shown to be independent of ZFC.

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Preliminaries

An R-module U is called *uniserial* if its submodules are totally ordered by inclusion. An integral domain R is a valuation domain if it is uniserial as an R-module. Throughout, R will denote a valuation domain and Q the quotient field of R; we assume $Q \neq R$. The uniserial modules we consider will always be generated by (at most) $\aleph_1$ elements, and most of the time R will be of cardinality $\aleph_1$. R^* denotes the set of units of R. The residue field of R is defined to be R/P, where P is the maximal ideal of R.

If J and A are R-submodules of Q with $A \subseteq J$, then J/A is a uniserial R-module, which is said to be *standard*. A uniserial R-module U is said to be *nonstandard* if it is not isomorphic to a standard uniserial.

Given a uniserial module U, and a nonzero element, a, of U, let $\text{Ann}(a) = \{r \in R: ra = 0\}$ and let $D(a) = \bigcup\{r^{-1}R: r$ divides a in $U\}$. We say U is of *type* J/A if $J/A \cong D(a)/\text{Ann}(a)$. This is well defined in that if b is another nonzero element of U, then $D(a)/\text{Ann}(a) \cong D(b)/\text{Ann}(b)$. For example, U has type Q/R if and only if U is divisible torsion and the annihilator ideal of every nonzero element of U is principal. (But notice that there is no $a \in U$ with $\text{Ann}(a) = R$.) It is not hard to see that if U has type J/A, then U is standard if and only if it is isomorphic to J/A.

Suppose U has type J/A and there exists $a \neq 0$ in U such that $A = \text{Ann}(a)$ and $J = D(A)$. Then

$$(*) \qquad\qquad J = \bigcup_{\nu < \omega_1} r_\nu^{-1}R$$

for some sequence of elements $\{r_\nu : \nu < \omega_1\}$ such that for all $\mu < \nu$, $r_\mu | r_\nu$. If J is countably generated, then U is standard, so generally we will be assuming that J is not countably generated; then it has a set of generators as in $(*)$, where, furthermore, r_ν does not divide r_μ if $\mu < \nu$, i.e., $r_\nu^{-1}r_\mu$ is not a member of R. Note that if $a_\nu \in U$ such that $r_\nu a_\nu = a$, then $\text{Ann}(a_\nu) = r_\nu A$,

so $a_\nu R \ (\subseteq U)$ is isomorphic to $R/r_\nu A = R/(A: r_\nu^{-1} R)$. If $\delta \in \lim(\omega_1)$ and

$$(**) \qquad J_\delta \overset{\text{def}}{=} \bigcup_{\nu < \delta} r_\nu^{-1} R$$

then the submodule of U generated by $\{a_\nu : \nu < \delta\}$ is a module over $R/(A : J_\delta) = R/\bigcap_{\nu < \delta} r_\nu A$.

Definition 1. A subset C of ω_1 is called a *cub*—short for closed unbounded set—if $\sup C = \omega_1$ and for all $Y \subseteq C$, $\sup Y \in \omega_1$ implies $\sup Y \in C$. Call two subsets, S_1 and S_2, of ω_1 equivalent iff there is a cub C such that $S_1 \cap C = S_2 \cap C$. Let $\widetilde{S}$ denote the equivalence class of S. The inclusion relation induces a partial order on the set, $D(\omega_1)$, of equivalence classes, i.e., $\widetilde{S_1} \le \widetilde{S_2}$ if and only if there is a cub C such that $S_1 \cap C \subseteq S_2 \cap C$. In fact, this induces a Boolean algebra structure on $D(\omega_1)$, with least element, 0, the equivalence class of sets disjoint from a cub; and greatest element, 1, the equivalence class of sets containing a cub. We say S is *stationary* if $\widetilde{S} \ne 0$, i.e., for every cub C, $C \cap S \ne \varnothing$. We say S is *costationary* if $\omega_1 \backslash S$ is stationary.

Given R and a type J/A, define $\Gamma_R(J/A)$ to be $\widetilde{S}$, where

$$S = \{\delta \in \lim(\omega_1): R/(A : J_\delta) \text{ is not complete}\}$$

where the topology on $R/(A : J_\delta)$ is the metrizable linear topology with a basis of neighborhoods of 0 given by the submodules $r_\nu A$ ($\nu < \delta$). $\Gamma_R(J/A)$ is well defined in the sense that it does not depend on the choice of the sequence $\{r_\nu : \nu < \omega_1\}$, because of the following, which is proved by a standard argument (cf. [EM, pp. 85ff]).

Lemma 2. *If* $J = \bigcup_{\nu < \omega_1} r_\nu^{-1} R$ *and also* $J = \bigcup_{\nu < \omega_1} s_\nu^{-1} R$, *then* $\{\delta: \bigcap_{\sigma < \delta} r_\sigma A = \bigcap_{\sigma < \delta} s_\sigma A\}$ *is a cub.*

Note that $\Gamma_R(J/A) = 0$ if J is countably generated.

If $J/A \cong J'/A'$, then $\Gamma_R(J/A) = \Gamma_R(J'/A')$, but in general $\Gamma_R(J/A)$ and $\Gamma_R(J'/A')$ will be different for different types J/A and J'/A'. For example, if R, A and J are as in Example 2 of [BS1, p. 302], then $\Gamma_R(J/A) = 1$, but $\Gamma_R(Q/A) = 0$ (since Q is countably generated). Note that if R is almost maximal, then $\Gamma_R(J/A) = 0$ for all types J/A.

In proving results about the existence or nonexistence of nonstandard uniserial modules it will be convenient to make use of the formulation of the problem given in [BS1], especially Lemma 1.2 and Test Lemma 1.3. Thus any uniserial module U of type J/A is described up to isomorphism by a family of units, $\{e_\sigma^\tau : \sigma < \tau < \omega_1\}$ such that

$$(\dagger) \qquad e_\tau^\delta e_\sigma^\tau - e_\sigma^\delta \in r_\sigma A$$

for all $\sigma < \tau < \delta < \omega_1$.

Moreover, if U is given by $(\dagger)$, then U is standard if and only if there exists a family $\{c_\sigma : \sigma < \omega_1\}$ of units of R such that

$$(\dagger\dagger) \qquad c_\tau - e_\sigma^\tau c_\sigma \in r_\sigma A$$

for all $\sigma < \tau < \omega_1$.

Salce has pointed out that by results of [BFS], the question of the existence of a nonstandard uniserial R-module of type J/A can be reduced to the question of the existence of a nonstandard uniserial of type K/R for an appropriate K.

THE CONJECTURE

We can paraphrase the conjecture of Osofsky in [O1, p. 164], for valuation domains R of cardinality $\aleph_1$, as follows:

R has a nonstandard uniserial of type J/A if and only if $\Gamma_R(J/A) = 1$.

Now, assuming CH (and the Axiom of Choice), it is possible to construct a valuation domain R of cardinality $\aleph_1$ such that $\Gamma_R(J/A) \neq 0, 1$ (see Definition 10 and the proof of Theorem 11). Thus the condition "$\Gamma_R(J/A) = 1$" cannot be necessary, because of the following result which implies that it is consistent with ZFC that there is a nonstandard uniserial R-module of type J/A for such an R. Note that $\diamondsuit_{\omega_1}(S)$, for any stationary S, is consistent with ZFC; in fact it is implied by $V = L$ (cf. [D, p. 139]).

Proposition 3. *Assume R has cardinality $\aleph_1$.*

(i) *If $\Gamma_R(J/A) = \widetilde{S}$, then $\diamondsuit_{\omega_1}(S)$ implies that there exists a nonstandard uniserial R-module of type J/A.*

(ii) *If $\Gamma(J/A) = 1$, then CH implies that there exists a nonstandard uniserial R-module of type J/A.*

Proposition 3(i) is essentially due to Fuchs and Salce [FS]; it can be proved, for example, by a straightforward generalization of the proof of Theorem 1.4 of [BS1], but for later purposes we will sketch the proof of a somewhat stronger result here. We begin with a lemma which, in one form or another, is a staple of the subject; its statement and proof, in our chosen notation, we include here for the sake of completeness.

Lemma 4. *Suppose $\{e_\sigma^\tau : \sigma < \tau < \delta\}$ is a family of units satisfying (†) and $R/\bigcap_{\nu<\delta} r_\nu A$ is not complete. Then there are units $e_{\sigma,j}^\delta$ ($\sigma < \delta$; $j = 0, 1$) such that*

$$e_{\tau,j}^\delta e_\sigma^\tau - e_{\sigma,j}^\delta \in r_\sigma A$$

for all $\sigma < \tau$ and $j \in \{0, 1\}$, but there is no sequence $\{c_\sigma : \sigma < \delta\}$ for which there are $c_{\delta,j} \in R^*$ ($j = 0, 1$) such that (††) holds for all $\sigma < \tau < \delta$ and moreover

$$c_{\delta,j} - e_{\sigma,j}^\delta c_\sigma \in r_\sigma A$$

for all $\sigma < \delta$ and $j \in \{0, 1\}$.

Proof. Let $e_{\sigma,0}^\delta$ be any family of units such that $e_{\sigma,0}^\delta - e_{\tau,0}^\delta e_\sigma^\tau \in r_\sigma A$ for all $\sigma < \delta$. Let $\langle v_\sigma : \sigma < \delta \rangle$ represent a Cauchy sequence in $R/\bigcap_{\nu<\delta} r_\nu A$ which does not have a limit in $R/\bigcap_{\nu<\delta} r_\nu A$. Let $e_{\sigma,1}^\delta = v_\sigma e_{\sigma,0}^\delta$. Then it is easy to check that $e_{\sigma,1}^\delta - e_{\tau,1}^\delta e_\sigma^\tau \in r_\sigma A$ for all $\sigma < \delta$. Moreover, if the conclusion of the lemma is contradicted by $\{c_\sigma : \sigma < \delta\}$ and $c_{\delta,0}, c_{\delta,1}$, then $c_{\delta,1} c_{\delta,0}^{-1}$ is a limit of $\langle v_\sigma : \sigma < \delta \rangle$, which is impossible. $\square$

Proof of Proposition 3 (sketch). In fact we will prove that, under the hypotheses on R, $\Phi_{\omega_1}(S)$ implies that there exists a nonstandard uniserial R-module of

type J/A. (This generalizes the result in [FrG].) Here $\Phi_{\omega_1}(S)$ is the weak diamond principle which is implied by $\Diamond_{\omega_1}(S)$ (see, for example, [EM, VI.1.6]). In [DSh] it is proved that CH implies $\Phi_{\omega_1}(\omega_1)$, so part (ii) follows.

Write R as the union, $R = \bigcup_{\nu < \omega_1} R_\nu$, of a continuous chain of countable subsets. For each $\delta \in S$, define a function P_δ on pairs (e, c) where $e = \{e_\sigma^\tau : \sigma < \tau < \delta\} \subseteq R_\delta$, $c = \{c_\sigma : \sigma < \delta\} \subseteq R_\delta$ as follows. If e satisfies (†), let $\{e_{\sigma,j}^\delta : \sigma < \delta\}$ ($j = 0, 1$) be as in Lemma 4; if e and c satisfy (††) and there is $c_0 \in R^*$ such that $c_0 - e_{\sigma,0}^\delta c_\sigma \in r_\sigma A$ for all $\sigma < \delta$, let $P_\delta(e, c) = 1$; in all other cases let $P_\delta(e, c) = 0$. Let $\rho: S \to 2$ be a weak diamond function (which predicts the values of the P_δ). Now define the e_σ^τ by induction. The crucial case is when $\delta \in S$ and $e = \{e_\sigma^\tau : \sigma < \tau < \delta\}$ is contained in R_δ; in this case let $e_\sigma^\delta = e_{\sigma,\rho(\delta)}^\delta$. After completing the inductive construction of the e_σ^τ for all $\sigma < \tau < \omega_1$, notice that for any sequence $\{c_\sigma : \sigma < \omega_1\}$, there is a cub C such that for all $\delta \in C$, $e_\sigma^\tau \in R_\delta$ and $c_\sigma \in R_\delta$ for all $\sigma < \tau < \delta$. Then the construction and the properties of the weak diamond function imply that there is no sequence $\{c_\sigma : \sigma < \omega_1\}$ satisfying (††). □

The condition $\Gamma_R(J/A) \neq 0$ is certainly necessary for the existence of a nonstandard uniserial R-module of type J/A, as the following easy lemma shows.

Lemma 5. *If* $\Gamma_R(J/A) = 0$, *then every uniserial* R-module of type J/A is standard.

Proof. Without loss of generality, we can assume that for all limit δ, $R/(A : J_\delta)$ is complete. Given $\{e_\sigma^\tau : \sigma < \tau < \omega_1\}$ as in (†), we define the units c_σ satisfying (††) by induction on σ. If c_σ has been defined, let $c_{\sigma+1} = e_\sigma^{\sigma+1} c_\sigma$. For a limit ordinal δ, consider the Cauchy sequence in $R/(A : J_\delta)$ represented by the elements $\langle e_\sigma^\delta c_\sigma : \sigma < \delta \rangle$; this sequence has a limit which is represented by a unit of R, which we call c_δ. □

Thus we end up with the following reformulated version of the conjecture:

($\$$) *For any valuation domain* R *of cardinality* $\aleph_1$, R *has a nonstandard uniserial module of type* J/A *if and only if* $\Gamma_R(J/A) \neq 0$.

Theorem 6. *Assuming* $V = L$, *this conjecture is true.*

Proof. In fact, this follows from Proposition 3 and Lemma 5 since Gödel's Axiom of Constructibility, $V = L$, implies $\Diamond_{\omega_1}(S)$ for all stationary S. □

Thus the conjecture is consistent with ZFC, and in fact with ZFC + GCH, since $V = L$ implies GCH.

ANOTHER GAMMA INVARIANT

Let R be a valuation domain of cardinality $\aleph_1$ and J, A as before. For any limit ordinal $\delta < \omega_1$, let

$$\mathscr{T}_{J/A}^\delta = \{\langle u_\sigma : \sigma < \delta \rangle : \forall \sigma < \tau < \delta (u_\sigma \in R^*, \text{ and } u_\tau - u_\sigma \in r_\sigma A)\};$$

that is, $\mathscr{T}_{J/A}^\delta$ consists of sequences of units which are Cauchy in the metrizable topology on $R/(A : J_\delta)$. Let $\mathscr{L}_{J/A}^\delta$ consist of those members of $\mathscr{T}_{J/A}^\delta$ which

have limits in R, i.e.,

$$\mathcal{L}_{J/A}^{\delta} = \{\langle u_{\sigma} : \sigma < \delta\rangle \in \mathcal{T}_{J/A}^{\delta} : \exists u_{\delta} \in R^{*} \text{ s.t. } \forall \sigma < \delta(u_{\delta} - u_{\sigma} \in r_{\sigma}A)\}.$$

Note that $\Gamma_{R}(J/A) = \widetilde{S}$ where

$$S = \{\delta \in \lim(\omega_{1}) : \mathcal{T}_{J/A}^{\delta} \neq \mathcal{L}_{J/A}^{\delta}\}.$$

We will be making use of an ω_{1}-*filtration of R by subrings*, by which we mean an increasing chain $\{N_{\alpha} : \alpha \in \omega_{1}\}$ of countable subrings of R such that $R = \bigcup_{\alpha \in \omega_{1}} N_{\alpha}$; and for limit α, $N_{\alpha} = \bigcup_{\beta < \alpha} N_{\beta}$.

Definition. $\Gamma'_{R}(J/A) = \widetilde{E}'$ where

$$E' = \{\delta \in \lim(\omega_{1}) : \exists \langle u_{\sigma} : \sigma < \delta\rangle \in \mathcal{T}_{J/A}^{\delta} \text{ s.t. } \forall \langle f_{\sigma} : \sigma < \delta\rangle \in \mathcal{L}_{J/A}^{\delta}$$
$$\exists \sigma < \delta \text{ s.t. } u_{\sigma}f_{\sigma} \notin N_{\delta}\}$$
$$= \{\delta \in \lim(\omega_{1}) : \exists \langle u_{\sigma} : \sigma < \delta\rangle \in \mathcal{T}_{J/A}^{\delta} \text{ s.t. } \forall f \in R^{*} \exists \sigma < \delta$$
$$\text{s.t. } u_{\sigma}f \notin N_{\delta} \bmod r_{\sigma}A\}.$$

As before, by standard methods it can be shown that the definition does not depend on the choice of $\{r_{\nu} : \nu < \omega_{1}\}$ or of $\{N_{\alpha} : \alpha < \omega_{1}\}$. (Here it is important that the N_{α} are countable.) Notice that $\Gamma'_{R}(J/A) \leq \Gamma_{R}(J/A)$ since if $\mathcal{T}_{J/A}^{\delta} = \mathcal{L}_{J/A}^{\delta}$, then we can let $\langle f_{\sigma} : \sigma < \delta\rangle$ be $\langle u_{\sigma}^{-1} : \sigma < \delta\rangle$ to show $\delta \notin E'$. We will prove below that the condition $\Gamma_{R}(J/A) \neq 0$ is not (provably from ZFC) sufficient for the existence of a nonstandard uniserial of type J/A; however, we do have the following theorem of ZFC.

Theorem 7. *If* $\Gamma'_{R}(J/A) \neq 0$, *then there is a nonstandard uniserial R-module.*

Proof. Let E' be as above. It may be helpful to regard the following as a $\Diamond(E')$-like argument carried out in ZFC, where the N_{δ} ($\delta \in E'$) serve to give the predictions. We will construct the e_{σ}^{τ} ($\sigma < \tau < \delta$) satisfying (†) by induction on δ so that for every $\delta \in E'$,

(#)   there is no sequence of units $\{c_{\sigma} : \sigma < \delta\} \subseteq N_{\delta}$ such that there exists $c_{\delta} \in R^{*}$ with $c_{\delta} - e_{\sigma}^{\delta}c_{\sigma} \in r_{\sigma}A$ for all $\sigma < \delta$.

Suppose we can do this. We claim that the uniserial module given by the e_{σ}^{τ} is nonstandard. Indeed if not, there exists $\{c_{\sigma} : \sigma < \omega_{1}\}$ as in (††); then there is a cub C such that for all $\delta \in C$, $c_{\sigma} \in N_{\delta}$ if $\sigma < \delta$; but then the construction is contradicted since $C \cap E'$ is nonempty.

The definition of the e_{σ}^{τ} is routine except for the case when $\delta \in E'$ and we are defining e_{σ}^{δ} assuming the e_{σ}^{τ} have been defined for $\sigma < \tau < \delta$. First choose any elements $\tilde{e}_{\sigma}^{\delta}$ so that $\{e_{\sigma}^{\tau} : \sigma < \tau < \delta\} \cup \{\tilde{e}_{\sigma}^{\delta} : \sigma < \delta\}$ satisfies (†). If (#) is true, then we can let $e_{\sigma}^{\delta} = \tilde{e}_{\sigma}^{\delta}$. Otherwise, we have $\langle d_{\sigma} : \sigma < \delta\rangle \subseteq N_{\delta}$ such that there exists $d_{\delta} \in R^{*}$ with

$$d_{\delta} - \tilde{e}_{\sigma}^{\delta} d_{\sigma} \in r_{\sigma}A \quad \text{for all } \sigma < \delta.$$

Let $\langle u_{\sigma} : \sigma < \delta\rangle$ be as in the definition of E' (i.e., there is no $f \in R^{*}$ such that for all $\sigma < \delta$, $u_{\sigma}f \in N_{\delta} \bmod r_{\sigma}A$). Let $e_{\sigma}^{\delta} = u_{\sigma}\tilde{e}_{\sigma}^{\delta}$. We claim (#) is satisfied. Suppose not; let $\{c_{\sigma} : \sigma < \delta\} \subseteq N_{\delta}$ such that there exists $c_{\delta} \in R^{*}$ with

$$c_{\delta} - e_{\sigma}^{\delta}c_{\sigma} \in r_{\sigma}A \quad \text{for all } \sigma < \delta.$$

Let $f = c_\delta^{-1} d_\delta$. Fix $\sigma < \delta$ and let $\equiv$ denote congruence mod $r_\sigma A$. Then we have $c_\delta \equiv e_\sigma^\delta c_\sigma \equiv u_\sigma \tilde{e}_\sigma^\delta c_\sigma$ and $d_\delta \equiv d_\sigma \tilde{e}_\sigma^\delta$, so $f \equiv u_\sigma^{-1} c_\sigma^{-1} d_\sigma$. Thus $u_\sigma f \equiv c_\sigma^{-1} d_\sigma \in N_\delta$, a contradiction of the choice of $\langle u_\sigma : \sigma < \delta \rangle$. $\square$

While the condition $\Gamma'_R(J/A) \neq 0$ is thus sufficient for there to be a nonstandard uniserial R-module of type J/A, we shall see later (Corollary 15) that it is not necessary.

Let us say that a type J/A is *essentially uncountable* if for every $\nu < \omega_1$ there exists $\mu > \nu$ such that $r_\nu A / r_\mu A$ is uncountable. This will be the case, for example, if R has uncountable residue field, J is not countably generated and $A = R$.

Theorem 8. ($\aleph_1 < 2^{\aleph_0}$) *If R is a valuation domain of cardinality $\aleph_1$ and J/A is essentially uncountable, then there is a nonstandard uniserial R-module of type J/A.*

Proof. It suffices to prove that $\Gamma'_R(J/A) \neq 0$; in fact we will show that $\Gamma'_R(J/A) = 1$. Without loss of generality we can assume $J = \bigcup_{\nu < \omega_1} r_\nu^{-1} R$ where for all $\nu < \omega_1$, $r_\nu A / r_{\nu+1} A$ has cardinality $\aleph_1$. Let $R = \bigcup_{\alpha < \omega_1} N_\alpha$ be an ω_1-filtration of R by subrings. Let δ be a limit ordinal. Choose a *ladder* on δ, i.e., a strictly increasing sequence γ_n ($n \in \omega$) whose limit is δ. We will define by induction on $n \in \omega$ a unit u_η for each $\eta \in {}^n 2$—the set of all functions from $n = \{0, 1, \ldots, n-1\}$ to $2 = \{0, 1\}$—such that $u_\eta - u_{\eta \restriction k} \in r_{\gamma_k} A$ for all $k < n$. If u_η has been defined and $\eta_i \in {}^{(n+1)} 2$ such that $\eta_i \restriction n = \eta$ and $\eta_i(n) = i$ ($i = 0, 1$), then we choose u_{η_i} congruent to u_η mod $r_{\gamma_n} A$ and such that $u_{\eta_1} u_{\eta_0}^{-1} \notin N_\delta$ mod $r_{\gamma_{n+1}} A$. This is possible since N_δ is countable and $r_{\gamma_n} A / r_{\gamma_{n+1}} A$ is uncountable. (Indeed, choose $a \in A$ such that $r_{\gamma_n} a \not\equiv y - 1$ mod $r_{\gamma_{n+1}} A$ for any $y \in N_\delta$; then let $u_{\eta_0} = u_\eta$ and $u_{\eta_1} = (1 + r_{\gamma_n} a) u_\eta$.) Thus for each of the $2^{\aleph_0}$ elements of ζ of ${}^\omega 2$, we have a different family $\langle u_{\zeta \restriction n} : n \in \omega \rangle$. We claim that for at least one ζ there is no $f \in R^*$ such that for all k, $u_{\zeta \restriction k} f \in N_\delta$ mod $r_{\gamma_k} A$—which will show $\delta \notin E'$. Indeed, if there were no such ζ, then since R^* has cardinality $\aleph_1 < 2^{\aleph_0}$, there would be $\zeta_0 \neq \zeta_1$ and f satisfying

$$u_{\zeta_j \restriction k} f \in N_\delta \quad \text{mod } r_{\gamma_k} A$$

for all $k \in \omega$ and $j \in \{0, 1\}$. Let n be such that $\zeta_0 \restriction n = \zeta_1 \restriction n$ but $\zeta_0(n) \neq \zeta_1(n)$. It is then easy to obtain a contradiction of our construction of u_{η_i} ($i = 0, 1$) if we let $\eta = \zeta_0 \restriction n$ and $\eta_i = \zeta_i \restriction (n+1)$. $\square$

Corollary 9. *Let R be a valuation domain of cardinality $\aleph_1$. If J/A is essentially uncountable and $\Gamma_R(J/A) = 1$, then there is a nonstandard uniserial R-module of type J/A.*

Proof. If CH holds, this is by Proposition 3; if CH fails this is by Theorem 8. $\square$

We will deal with the essentially countable case in a later paper; in fact, Corollary 9 can be proved without the hypothesis of essential uncountability, but this requires a different construction in the essentially countable case. This

means that Osofsky's original conjecture is vindicated for "natural" valuation domains, where $\Gamma_R(J/A)$ has to be either 0 or 1.

A CONSTRUCTION OF NONSTANDARD UNISERIALS

As an application of Theorem 7, we will show how to construct nonstandard uniserials (of type Q/R) in ZFC over certain concretely given valuation domains R. (This may be regarded as a generalization of [O2] in that we define such R with any given nonzero value for $\Gamma_R(Q/R)$.) We borrow an idea from [O2] (the definition of u_σ) in order to show that $\Gamma'_R(Q/R) > 0$. First we define the ring.

Definition 10. Let G be the ordered abelian group which is the direct sum $\bigoplus_{\alpha < \omega_1} \mathbb{Z}\alpha$ ordered antilexicographically; that is, $\sum_\alpha n_\alpha \alpha > 0$ if and only if $n_\beta > 0$, where β is maximal such that $n_\beta \neq 0$. (This is the group used in [FS] and [O2] and denoted $\Gamma(\omega_1)$ in [BS1, §3]). In particular, the basis elements have their natural order and if $\alpha < \beta$, then $k\alpha < \beta$ in G for any $k \in \mathbb{Z}$.) Let $G^+ = \{g \in G : g \geq 0\}$ and for each $\gamma < \omega_1$, let G_γ^+ be the submonoid $\{g \in G : 0 \leq g < \gamma\}$ of G^+.

Fix a field K of cardinality $\leq \aleph_1$ and let

$$\widehat{R} = \left\{ \sum_{y \in \Delta} k_g X^g : k_g \in K, \ \Delta \ \text{a well-ordered subset of } G^+ \right\},$$

$(= K[[G]]$; cf. [FS, pp. 24ff])

$$\widehat{R}_{\alpha+1} = \left\{ \sum_{g \in \Delta} k_g X^g \in \widehat{R} : \Delta \subseteq G_\alpha^+ \right\}$$

and for γ a limit ordinal, $\widehat{R}_\gamma = \bigcup_{\alpha < \gamma} \widehat{R}_\alpha$.

Given an element $y = \sum_{g \in \Delta} k_g X^g$ of $\widehat{R}$, let $\mathrm{supp}(y) = \{g \in \Delta : k_g \neq 0\}$; let $p\text{-}\mathrm{supp}(y) = \{\alpha \in \omega_1 : \exists g \in \mathrm{supp}(y)$ whose projection on $\mathbb{Z}\alpha$ is nonzero$\}$. If $X \subseteq G$, then $y \upharpoonright X$ is defined to be $\sum_{g \in X \cap \Delta} k_g X^g$. Let $y \upharpoonright \alpha = y \upharpoonright \{g \in G : g < \alpha\}$, the *truncation* of y to α. Note that $y \in \widehat{R}_{\alpha+1}$ if and only if $y = y \upharpoonright \alpha$.

For $Y \subseteq \widehat{R}$, let $\langle Y \rangle_{\mathrm{val}}$ denote the valuation subring generated by Y, i.e., the intersection with $\widehat{R}$ of the quotient field of the subring generated by Y.

For any stationary subset S of ω_1, let R_s be the subring of $\widehat{R}$ consisting of all elements y of $\widetilde{R}$ such that $\mathrm{supp}(y)$ is countable and no member of S is a limit point of $p\text{-}\mathrm{supp}(y)$ (in the order topology).

Then R_S is a valuation domain since $p\text{-}\mathrm{supp}(xy^{-1}) \subseteq p\text{-}\mathrm{supp}(x) \cup p\text{-}\mathrm{supp}(y)$. The cardinality of R_S is $\leq 2^{\aleph_0}$.

Theorem 11. (CH) *For every field K of cardinality $\aleph_1$ and every stationary subset S of ω_1, there is a valuation domain R of cardinality $\aleph_1$ with residue field K and quotient field Q such that $\Gamma_R(Q/R) = \widetilde{S}$ and there is a nonstandard uniserial R-module of type Q/R.*

Proof. Let R be R_S as defined above. Let $r_\sigma = X^\sigma$, so that $Q = \bigcup_{\sigma < \omega_1} r_\sigma^{-1} R$. First notice that $\Gamma_R(Q/R) \geq \widetilde{S}$ since for any $\delta \in S \cap \lim(\omega_1)$, if γ_n $(n \in \omega)$

is a ladder on δ and y_n is defined to be $\sum_{i=0}^{n-1} X^{\gamma_i}$ then $\{y_n : n < \omega\}$ is a Cauchy sequence in $R/(R : J_\delta)$ which does not have a limit, since a putative limit has δ as a limit point of its p-support. Moreover, $\Gamma_R(Q/R) \leq \widetilde{S}$ since for $\delta \notin S$, a Cauchy sequence in $R/(R : J_\delta)$ *will* have a limit in R because δ is allowed to be the limit point of the p-support of an element of R.

To show that there is a nonstandard uniserial of type Q/R, it suffices by Theorem 7 to show that $\Gamma'_R(Q/R) \geq \widetilde{S}$. Let us fix an ω_1-filtration of R by subrings, $\{N_\alpha : \alpha \in \omega_1\}$. Let $\delta \in S$ and let $\langle \gamma_n : n \in \omega \rangle$ be a ladder on δ. Let L be the subfield of K generated by the coefficients of the members of N_δ. Then L is a countable subfield of K so we can inductively choose $t_n \in K \backslash L(t_0, \ldots, t_{n-1})$. Let $u_\sigma = 1 + \sum_{i=0}^{n-1} t_i X^{\gamma_i}$ for all σ with $\gamma_{n-1} < \sigma \leq \gamma_n$. Then clearly $\langle u_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{Q/R}^\delta$. It suffices to show that there is no $f \in R^*$ such that for all $\sigma < \delta$, $u_\sigma f \in N_\delta$ mod $X^\sigma R$. Suppose there is such an f; by definition of R_S and since $\delta \in S$, there exists n such that $\sup(p\text{-}\mathrm{supp}(f \restriction \delta)) \leq \gamma_{n-2}$. Say $f \restriction \delta = \sum_{g \leq \gamma_{n-2}} k_g X^g$. But then, because of the properties of the value group—in particular because $2\gamma_{n-2} < \gamma_{n-1}$—we see that the coefficient of $X^{\gamma_{n-1}}$ in $u_{\gamma_n} f$ is $t_{n-1} k_0$; similarly the coefficient of X^{γ_n} in $u_{\gamma_{n+1}} f$ is $t_n k_0$. By assumption, these coefficients belong to L, and we easily obtain a contradiction of the choice of the t_i. $\square$

Remarks. (1) The hypothesis of CH was used only to insure that R_S has cardinality $\aleph_1$. The ring defined by Osofsky in [O2] has cardinality $\aleph_1$ without invoking CH, and the proof above applies to it. (It satisfies $\Gamma'(Q/R) = 1$.) Bazzoni has pointed out that the proof of Theorem 7 can be adapted to prove Theorem 11 for the rings R_S, even when R_S has cardinality $2^{\aleph_0} > \aleph_1$ and so cannot have an ω_1-filtration by countable subrings: choose an ω_1-filtration of K by countable subfields L_α and define N_α to be the subrings of elements whose coefficients come from L_α.

(2) The main result of [O1] is very general in that it deals with arbitrary types, not just Q/R. Similarly the method of proof of Theorem 11 can be used to prove, in ZFC, that there exist nonstandard uniserials of many other types than Q/R, e.g., ones like those given in Examples 1–6 of [BS1, pp. 301–305].

AXIOM(S)

As we have observed, Lemma 5 says that the condition $\Gamma_R(J/A) \neq 0$ is necessary for the existence of a nonstandard uniserial of type J/A. We now aim to prove that the condition $\Gamma_R(J/A) \neq 0$ is *not* provably sufficient if we assume only GCH (rather than V = L), i.e., the "if" direction of the conjecture fails in some model of ZFC + GCH. By Proposition 3 and Theorem 7, we will need to consider R and J/A where $\Gamma'_R(J/A) = 0$ and $\Gamma_R(J/A) \neq 1$. Analogies with the Whitehead Problem suggest the use of the following principle, known as Ax(S), which has been shown by the second author to be consistent with ZFC + GCH (and with ZFC + ¬CH):

> there is a stationary and costationary subset S of ω_1 such that for all proper posets $\mathbb{P}$ of cardinality $\aleph_1$ which are $(\omega_1 \backslash S)$-complete, and for all families $\mathscr{D}$ of $\aleph_1$ dense subsets of $\mathbb{P}$, there is a $\mathscr{D}$-generic subset of $\mathbb{P}$.

(See [EM, pp. 170–173] for the necessary definitions.)

We will be able to prove that in a model of $\mathrm{Ax}(S) + \mathrm{CH}$, there exists a valuation domain R of cardinality $\aleph_1$ with $\Gamma_R(Q/R) = \tilde{S}$ such that every uniserial of type Q/R is standard.

Definition 12. For any countable subring N of a valuation domain R of cardinality $\aleph_1$, let

$$\mathscr{T}_{J/A}^{\delta}(N) = \{\langle \overline{u}_\sigma : \sigma < \delta \rangle : \forall \sigma < \tau < \delta (u_\sigma \in N \text{ and } u_\tau - u_\sigma \in r_\sigma A)\}$$

where $\overline{u}_\sigma$ denotes the coset $u_\sigma + r_\sigma A$. Thus $\mathscr{T}_{J/A}^{\delta}(N)$ consists of sequences $\langle \overline{u}_\sigma : \sigma < \delta \rangle$ where $\langle u_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{J/A}^{\delta}$ and all the u_σ belong to N. Usually we shall abuse notation and write $\langle u_\sigma : \sigma < \delta \rangle$ for an element of $\mathscr{T}_{J/A}^{\delta}(N)$; but notice that $\overline{u}_\tau$ determines $\overline{u}_\sigma$ for all $\sigma < \tau$. Let

$$\mathscr{L}_{J/A}^{\delta}(N) = \{\langle \overline{u}_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{J/A}^{\delta}(N) : \langle u_\sigma : \sigma < \delta \rangle \in \mathscr{L}_{J/A}^{\delta}\}.$$

$\mathscr{T}_{J/A}^{\delta}(N)$ is given the tree topology, i.e., if $u = \langle \overline{u}_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{J/A}^{\delta}(N)$, a basis of neighborhoods of u is the family $\{[u]_\nu : \nu < \delta\}$, where $[u]_\nu = \{\langle \overline{v}_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{J/A}^{\delta}(N) : \overline{v}_\sigma = \overline{u}_\sigma \text{ for all } \sigma \leq \nu\}$.

From now on S always denotes the stationary and costationary subset of ω_1 which is asserted to exist by $\mathrm{Ax}(S)$.

Theorem 13. ($\mathrm{Ax}(S)$) *Suppose that R is a valuation domain of cardinality $\aleph_1$ such that $\Gamma_R(J/A) \subseteq \tilde{S}$ and $\Gamma'_R(J/A) = 0$. Suppose in addition that for some ω_1-filtration of R by subrings, $R = \bigcup_{\alpha \in \omega_1} N_\alpha$, the following holds for all $\delta \in S$:*

(&)   *for every open subset U of $\mathscr{T}_{J/A}^{\delta}(N_\delta)$, $U \cap \mathscr{L}_{J/A}^{\delta}(N_\delta)$ is a non-meager (i.e., second category) subset of $\mathscr{T}_{J/A}^{\delta}(N_\delta)$.*

Then every uniserial R-module of type J/A is standard.

Proof. Let U be a uniserial R-module of type J/A given by a family $\{e_\sigma^\tau : \sigma < \tau < \omega_1\}$ as in (†). Let the poset $\mathbb{P}$ consist of all sequences $p = \langle \overline{p}_\sigma : \sigma \leq \mu \rangle$ where μ is a countable ordinal (denoted $l(p)$ and called the *length* of p) such that

$$\forall \sigma < \tau \leq \mu (p_\tau \in R^*, \text{ and } p_\tau - e_\sigma^\tau p_\sigma \in r_\sigma A).$$

The partial ordering is the natural one of extension of sequences. In an abuse of notation we shall write the elements of $\mathbb{P}$ as if they were sequences of elements of R rather than sequences of cosets. Note that $\mathbb{P}$ is a tree and has the following properties:

(i) if $p, q \in \mathbb{P}$, $p_\tau = q_\tau$ for some τ and c is a sequence of length $= l(q) \geq \tau$, such that $c_\sigma \equiv p_\sigma \pmod{r_\sigma A}$ for all $\sigma \leq \tau$, and $c_\sigma \equiv q_\sigma \pmod{r_\sigma A}$ for all $\tau \leq \sigma \leq l(q)$, then $c \in \mathbb{P}$;

(ii) if $p \in \mathbb{P}$ and $e \in R^*$, then $\langle ep_\sigma : \sigma \leq l(p) \rangle \in \mathbb{P}$.

For each $\nu < \omega_1$, let $D_\nu = \{p \in \mathbb{P} : p \text{ has length } \geq \nu\}$. Then D_ν is dense in $\mathbb{P}$ because given $q = \langle q_\sigma : \sigma \leq \mu \rangle \in \mathbb{P}$ (where we can assume $\mu < \nu$), define $p_\sigma = q_\sigma$ if $\sigma \leq \mu$ and $p_\sigma = (e_\mu^\nu)^{-1} e_\mu^\nu q_\mu$ if $\mu \leq \sigma \leq \nu$. Then if $\sigma < \tau \leq \nu$, one may easily check that modulo $r_\sigma A$,

$$p_\tau - e_\sigma^\tau p_\sigma \equiv ((e_\tau^\nu)^{-1} - e_\sigma^\tau (e_\sigma^\nu)^{-1}) e_\mu^\nu q_\mu \equiv 0 \cdot e_\mu^\nu q_\mu$$

if $\mu \leq \sigma < \tau \leq \nu$, and similarly if $\sigma < \mu < \tau \leq \nu$. Thus $\langle p_\sigma : \sigma \leq \nu \rangle$ is an element of $\mathbb{P}$ extending q, so D_ν is dense in $\mathbb{P}$.

It suffices then to prove that there is a $\{D_\nu : \nu < \omega_1\}$-generic subset of $\mathbb{P}$; and for this, given the set-theoretic hypothesis, it suffices to prove that $\mathbb{P}$ is an $(\omega_1 \backslash S)$-complete proper poset.

Let κ be large enough for $\mathbb{P}$; let $\mathscr{C}$ be the cub of all countable elementary submodels N of $(H(\kappa), \in, \omega_1, \{e_\sigma^\tau : \sigma < \tau < \omega_1\}, R, \ldots)$ such that $N = \bigcup_{i \in \omega} M_i$ where $M_i \prec M_{i+1}$ and $M_i \cap \omega_1 < M_{i+1} \cap \omega_1$. Fix $N \in \mathscr{C}$ and let $\delta = N \cap \omega_1$, $\delta_i = M_i \cap \omega_1$. To show that $\mathbb{P}$ is $(\omega_1 \backslash S)$-complete consider a chain $p^0 \leq p^1 \leq \cdots$ of elements $p^n = \langle p_\sigma^n : \sigma \leq l(p^n) \rangle$ of $\mathbb{P} \cap N$ such that for each dense subset D of $\mathbb{P}$ which belongs to N there exists n with $p^n \in D$. For all n there exists m_n such that $l(p^{m_n}) \geq \delta_n$ (because $D_{\delta_n} \in N_{n+1}$). For each $\sigma < \delta$, let $p_\sigma = p_\sigma^m$ for any m such that $l(p^m) \geq \sigma$. Then $\{e_0^\delta p_\sigma : \sigma < \delta\}$ is a Cauchy sequence in $R/(A : J_\delta)$, so if $\delta \notin S$, it has a limit $p_\delta \in R$, and then $\langle p_\sigma : \sigma \leq \delta \rangle$ is a member of $\mathbb{P}$ which extends each p^n.

So far, we have used just the hypothesis that $\Gamma_R(J/A) \leq \tilde{S}$. To show that $\mathbb{P}$ is proper, we need to use the additional hypotheses on R. Since the intersection of two cubs in a cub, we can assume, without loss of generality, that $N \cap R = N_\delta$ (the δth member of the ω_1-filtration of R given in the statement of the theorem).

Let $q \in \mathbb{P} \cap N$; we must find $p \geq q$ such that p is N-generic. This is no problem if $\delta \notin S$ since $\mathbb{P}$ is $(\omega \backslash S)$-complete, so assume $\delta \in S$. Choose $d = \langle d_\sigma : \sigma \leq \delta \rangle \in \mathbb{P}$ of length δ and let $u_\sigma = e_0^\delta (e_0^\sigma)^{-1} d_\sigma$. Then $\langle u_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{J/A}^\delta$. Since $\Gamma_R'(J/A) = 0$ we can assume that there exists $\langle f_\sigma : \sigma < \delta \rangle \in \mathscr{L}_{J/A}^\delta$ with limit f_δ so that $u_\sigma f_\sigma \in N_\delta$ for all $\sigma < \delta$. Then, replacing d by $\langle e_0^\delta d_\sigma f_\sigma : \sigma \leq \delta \rangle$, we can assume that d_σ belongs to N_δ for all $\sigma < \delta$ (since $e_0^\delta d_\sigma f_\sigma = e_0^\sigma u_\sigma f_\sigma$; note that $e_0^\sigma \in N_\delta$ for all $\sigma < \delta$). Also, by (i) and (ii), we can assume that $d_\sigma = q_\sigma$ for all $\sigma \leq l(q)$. (Note that $l(q) < \delta$ since $q \in N$ and $N \cap \omega_1 = \delta$.)

Let U be the open subset

$$\{\langle u_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{J/A}^\delta (N_\delta) : u_\sigma = 1 \text{ for } \sigma \leq l(q)\}.$$

Given an element $u = \langle u_\sigma : \sigma < \delta \rangle$ of $\mathscr{T}_{J/A}^\delta (N_\delta)$, we will let du denote $\langle d_\sigma u_\sigma : \sigma < \delta \rangle$; notice that for all $\sigma < \tau < \delta$ we have

$$d_\tau u_\tau - e_\sigma^\tau d_\sigma u_\sigma = d_\tau u_\tau - d_\tau u_\sigma + d_\tau u_\sigma - e_\sigma^\tau d_\sigma u_\sigma \in r_\sigma A,$$

so du belongs to $\mathbb{P}$. For each dense subset D of $\mathbb{P}$ which belongs to N, let

$$D' = \{u \in \mathscr{T}_{J/A}^\delta (N_\delta) : du \text{ is not an extension of a member of } D \cap N\}.$$

Then D' is a nowhere dense subset of $\mathscr{T}_{J/A}^\delta (N_\delta)$ since for any basic open subset $[u]_\nu$, we can choose $u' = \langle u_\sigma' : \sigma \leq l(u') \rangle \geq \langle d_\sigma u_\sigma : \sigma \leq \nu \rangle$ which belongs to $D \cap N$, and then $\langle d_\sigma^{-1} u_\sigma' : \sigma \leq l(u') \rangle$ determines a basic nonempty open subset of $[u]_\nu$ which is disjoint from D'. (We are using here the fact that in the elementary submodule N, $D \cap N$ is a dense subset of $\mathbb{P} \cap N$; moreover note that $\langle \overline{d_\sigma u_\sigma} : \sigma \leq \nu \rangle$ belongs to $\mathbb{P} \cap N$.) Hence by (&), there is an element $w = \langle w_\sigma : \sigma < \delta \rangle$ of $U \cap \mathscr{L}_{J/A}^\delta$ which does not belong to D' for any $D \in \mathbb{P} \cap N$. Let w_δ be such that $w_\delta - w_\sigma \in r_\sigma A$ for all $\sigma < \delta$. Then $\langle d_\sigma w_\sigma : \sigma \leq \delta \rangle$ is the desired N-generic element extending q. $\square$

Now the following result shows that the conjecture ($) fails in a model of $Ax(S) + CH$.

Theorem 14. $(Ax(S) + CH)$ *For any field K of cardinality $\leq \aleph_1$, there is a valuation domain R of cardinality $\aleph_1$ with residue field K and a stationary, costationary subset S of ω_1 such that $\Gamma_R(Q/R) = \tilde{S}$, but every uniserial R-module of type Q/R is standard.*

Proof. Let $\hat{R}$ be as in Definition 10; it suffices to show there is a valuation subring R of $\hat{R}$ of cardinality $\aleph_1$ which satisfies the hypotheses of Theorem 13 for the type Q/R. We let $r_\nu = X^\nu$ which will be in R. (Thus Q will be $\bigcup_{\nu < \omega_1} r_\nu^{-1} R$.) For each $\gamma \in \omega_1$, let $W_\gamma = \{y \in \hat{R}_\gamma : \operatorname{supp}(y)$ is finite and the coefficients of y belong to the prime subfield of $K\}$.

We will now define by induction on $\alpha < \omega_1$ a continuous chain $\{Y_\alpha : \alpha < \omega_1\}$ of valuation subrings of $\hat{R}$ such that for all α: $W_\alpha \subseteq Y_\alpha \subseteq \hat{R}_\alpha$; $Y_\alpha \cap R^* = Y_\alpha^*$. Moreover, the Y_α will be closed under truncation, that is, if $y \in Y_\alpha$ and $\beta < \alpha$, then $y \restriction \beta \in Y_{\beta+1}$. We will then define R to be the union of the Y_α.

First of all notice that each element of $\mathscr{T}_{Q/R}^\delta(N)$ is uniquely represented by a sequence $u = \langle u_\sigma : \sigma < \delta \rangle$ where for all $\sigma < \delta$, $u_\sigma \in N$, and $u_\tau \restriction \sigma = u_\sigma$ whenever $\tau \geq \sigma$. We shall consider these to be the elements of $\mathscr{T}_{Q/R}^\delta(N)$. Moreover, given $u = \langle u_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{Q/R}^\delta(Y_\delta)$, there is a unique element of $\hat{R}_{\delta+1}$, which we will denote u_δ, which represents the limit of $\langle u_\sigma : \sigma < \delta \rangle$ in $\hat{R}/\bigcap_{\nu<\delta} r_\nu \hat{R}$. If $u_\delta \notin Y_{\delta+1}$, then, by closure under truncation, the sequence $\langle u_\sigma : \sigma < \delta \rangle$ will not have a limit in R. Given a subset Z of $\mathscr{T}_{Q/R}^\delta(Y_\delta)$, we will denote by $Z_{[\delta]}$ the set $\{u_\delta : u \in Z\} \subseteq \hat{R}_{\delta+1}$.

Let $Y_0 = \langle\{1\}\rangle_{\text{val}}$. Suppose that Y_β has been defined for all $\beta < \gamma$. If γ is a limit ordinal, let $Y_\gamma = \bigcup_{\beta<\gamma} Y_\beta$. If $\gamma = \delta + 1$ and δ is a successor ordinal, let Y_γ be the valuation subring of $\hat{R}$ generated by $Y_\delta \cup W_{\delta+1}$. If $\delta \notin S$, let Y_γ be the valuation subring generated by Y_δ together with the u_δ for each $u \in \mathscr{T}_{Q/R}^\delta(Y_\delta)$.

If δ is a limit ordinal in S, we must satisfy the hypotheses of Theorem 13 and make sure $R/\bigcap_{\nu<\delta} r_\nu R$ is not complete. Let $\{(N_\beta, \Phi_\beta, V_\beta): \beta < 2^{\aleph_0} = \aleph_1\}$ be an enumeration of all triples such that N_β is a countable subring of Y_δ, V_β is a basic open set in $\mathscr{T}_{Q/R}^\delta(N_\beta)$, and Φ_β is the complement of a countable union of closed nowhere dense subsets of $\mathscr{T}_{Q/R}^\delta(N_\beta)$. (This enumeration is possible since each $\mathscr{T}_{Q/R}^\delta(N_\beta)$ has a countable base of open sets.) Also enumerate $\mathscr{T}_{Q/R}^\delta(Y_\delta)$ as $\{u^\alpha = \langle u_\sigma^\alpha : \sigma < \delta \rangle : \alpha < 2^{\aleph_0}\}$. We will inductively define elements y_α of $\hat{R}_{\delta+1}$ $(\alpha < 2^{\aleph_0})$ and then let $Y_{\delta+1} = \langle Y_\delta \cup \{y_\alpha : \alpha < 2^{\aleph_0}\}\rangle_{\text{val}}$. Fix an element $c = \langle c_\sigma : \sigma < \delta \rangle$ of $\mathscr{T}_{Q/R}^\delta(Y_\delta)$ and let $c_\delta \in \hat{R}_{\delta+1}$ be its limit, as above. We also require that c is chosen so that $c_\delta \notin \hat{R}_\delta$. The elements y_α will be chosen with the following properties for all $\alpha < 2^{\aleph_0}$:

(i) if $\alpha = 2\beta + 1$, then $y_\alpha = u_\delta$ for some $u \in \Phi_\beta \cap V_\beta$;

(ii) if $\alpha = 2\beta$, then $y_\alpha = f_\delta$ for some $f = \langle f_\sigma : \sigma < \delta \rangle \in \mathscr{T}_{Q/R}^\delta(Y_\delta)$ such that $u_\sigma^\beta f_\sigma \in W_\delta$ for all $\sigma < \delta$;

(iii) $c_\delta \notin \langle Y_\delta \cup \{y_\gamma : \gamma < \alpha\}\rangle_{\text{val}}$.

Then (i) (resp. (ii)) will imply that (&) holds (resp. $\Gamma_R'(Q/R) = 0$), and (iii) will insure that $R/\bigcap_{\nu < \delta} r_\nu R$ is not complete.

Suppose that y_γ has been defined for all $\gamma < \alpha$ and that $\alpha = 2\beta + 1$. Temporarily let $R' = \langle Y_\delta \cup \{y_\gamma : \gamma < \alpha\}\rangle_{\mathrm{val}}$. We claim that

(14.1)   there is a subset Z of $\Phi_\beta \cap V_\beta$ of cardinality > 1 such that $Z_{[\delta]}$ is algebraically independent over R'.

If so, there exists $u \in Z$ such that $c_\delta \notin \langle R' \cup \{u_\delta\}\rangle_{\mathrm{val}}$ and then we can let $y_\alpha = u_\delta$. For otherwise, we have $u \neq u'$ in Z and nonzero polynomials $f_1(T), f_2(T), g_1(T), g_2(T) \in R'[T]$ such that $c_\delta = f_1(u_\delta)/g_1(u_\delta)$ and $c_\delta = f_2(u_\delta')/g_2(u_\delta')$. But then the polynomial $f_1(T_1)g_2(T_2) - f_2(T_2)g_1(T_1) \in R'[T_1, T_2]$ is nonzero (because $c_\delta \notin R'$) and shows that $\{u_\delta, u_\delta'\}$—and hence $Z_{[\delta]}$—is not algebraically independent over R'. To prove (14.1), we show

(14.2)   there is a subset Z' of $\Phi_\beta \cap V_\beta$ of cardinality $2^{\aleph_0}$ such that $Z'_{[\delta]}$ is algebraically independent over Y_δ.

This suffices, by the additivity of transcendence degree, since $|\{y_\gamma : \gamma < \alpha\}| < 2^{\aleph_0}$.

In order to prove (14.2), we write the complement of Φ_β as the union, $\bigcup_{n \in \omega} C_n$, of countably many closed nowhere dense sets in $\mathscr{T}_{Q/R}^\delta(N_\beta)$. Choose a ladder $\{\tau_n : n \in \omega\}$ on δ and inductively define a tree of basic clopen subsets $U_\eta \subseteq V_\beta$ ($\eta \in {}^n 2$) so that $U_\eta \subseteq U_\zeta$ if $\zeta = \eta \restriction l(\zeta)$ and $U_\eta \subseteq$ the complement of $C_{l(\eta)}$. Moreover, we choose the U_η so that for each n, if ${}^n 2 = \{\eta_1, \ldots, \eta_{2^n}\}$, then there are ordinals γ_i with $\tau_n < \gamma_0 < \cdots < \gamma_{2^n}$ such that for every $z_i \in U_{\eta_i}$ ($i = 1, \ldots, 2^n$) and $g \in G^+$, if $\gamma_1 \leq g \leq \gamma_{2^n}$, then $g \in \mathrm{supp}(z_i)$ if and only if $g = \gamma_{i-1}$. Then one may prove by induction on m that if $f(T_1, \ldots, T_m) \in Y_\delta[T_1, \ldots, T_m]$ is a nonzero polynomial whose coefficients are in $\widehat{R}_{\tau_n}$ and if $z_j \in U_{\eta_j}$, then $f(z_1, \ldots, z_m) \in \widehat{R}_{\gamma_{m+1}} \setminus \{0\}$. Hence if we choose for each $\xi \in {}^\omega 2$, $z_\xi \in \bigcap_{n \in \omega} U_{\xi \restriction n}$, then $Z_{[\delta]} = \{z_\xi : \xi \in {}^\omega 2\}$ is algebraically independent over Y_δ.

Finally consider the case when $\alpha = 2\beta$. It suffices to show that there is a subset Z of $\mathscr{T}_{Q/R}^\delta(Y_\delta)$ of cardinality $2^{\aleph_0}$ so that $Z_{[\delta]}$ is algebraically independent over Y_δ and each $f \in Z$ has the form $\langle w_\sigma(u_\sigma^\beta)^{-1} : \sigma < \delta\rangle$ where $\langle w_\sigma : \sigma < \delta\rangle \in \mathscr{T}_{Q/R}^\delta(Y_\delta)$ and for each $\sigma < \delta$, $w_\sigma \in W_\delta$. Now by a tree argument similar to that above, there is a subset Z' of $\mathscr{T}_{Q/R}^\delta(Y_\delta)$ of cardinality $2^{\aleph_0}$ consisting of elements $w = \langle w_\sigma : \sigma < \delta\rangle$ such that for all σ the coefficients of w_σ belong to $\{0, 1\}$ and such that $Z'_{[\delta]}$ is algebraically independent over Y_δ. Then by an argument on transcendence degree, there is a subset S of $\{w_\delta(u^\beta)_\delta^{-1} : w \in Z'\}$ of cardinality $2^{\aleph_0}$ which is algebraically independent over Y_δ. So let

$$Z = \{\langle w_\sigma(u_\sigma^\beta)^{-1} : \sigma < \delta\rangle : w_\delta(u^\beta)_\delta^{-1} \in S\}. \quad \square$$

The following corollary shows that the converse of Theorem 7 is not provable in ZFC + GCH.

Corollary 15. (V = L) *There is a valuation domain R of cardinality $\aleph_1$ such that $\Gamma_R'(Q/R) = 0$ and there is a nonstandard uniserial R-module of type Q/R.*

Proof. Let R be the valuation domain constructed as in the proof of Theorem 14 for some stationary S. (The construction requires only CH.) Then $\Gamma'_R(Q/R) = 0$. Also $\Gamma_R(Q/R) = \tilde{S} \neq 0$, so by Proposition 3, there is a nonstandard uniserial R-module, since $V = L$, and hence $\Diamond_{\omega_1}(S)$ holds. $\square$

VALUATION DOMAINS OF CARDINALITY $> \aleph_1$

Finally we want to observe that the hypothesis that R has cardinality $\aleph_1$ is essential for Proposition 3; that is, it is not enough to require that J is generated by $\aleph_1$ elements (in which case the definition of $\Gamma_R(J/A)$ makes sense). In fact, we have the following theorem of ZFC.

Theorem 16. *There is a valuation domain R of cardinality $2^{\aleph_1}$ such that Q is generated (as R-module) by $\aleph_1$ elements and $\Gamma_R(Q/R) = 1$, but every uniserial R-module of type Q/R is standard.*

Proof. Let G be the free abelian group on $\omega_1 \times 2^{\aleph_1}$. We order $\omega_1 \times 2^{\aleph_1}$ lexicographically and then order G antilexicograhically (cf. Definition 10). Thus in G, $(\alpha, i) < (\beta, j)$ if and only if $\alpha < \beta$ (in ω_1) or $\alpha = \beta$ and $i < j$ (in $2^{\aleph_1}$) and in that case $n(\alpha, i) < (\beta, j)$ for all $n \in \mathbb{Z}$. Let $\widehat{R} = K[[G]]$ be defined as in Definition 10, for this G and for a field K of cardinality $\leq \aleph_1$. Define p-supp and truncation analogously to Definition 10.

Let Y_1 be the smallest valuation subring of $\widehat{R}$ containing $X^{(\alpha, 0)}$ for all $\alpha \in \omega_1$ and let $r_\nu = X^{(\nu, 0)}$ for $\nu < \omega_1$. Then $Q = \bigcup_{\nu < \omega_1} r_\nu^{-1} R$ and $\Gamma_{Y_1}(Q/Y_1) = 1$.

Define G_i to be the subgroup of G generated by $\{(\alpha, j) : \alpha < \omega_1, \ j < i\}$ and let $\widehat{R}_i$ be the subring of $\widehat{R}$ consisting of all $y \in \widehat{R}$ such that $\text{supp}(y) \subseteq G_i$. We shall now define a continuous chain $\{Y_i : i < 2^{\aleph_1}\}$ of valuation subrings of R such that for each i, $Y_i \subseteq \widehat{R}_i$, and is closed under truncation—more precisely, if $y \in Y_i$ and $\delta \in \lim(\omega_1)$, then $y \upharpoonright (\delta, 0) \in Y_i$. Moreover, we require that

$$(\pounds) \qquad\qquad \text{for all } i < j, \qquad Y_j \cap \widehat{R}_i = Y_i.$$

We will let R be the union of the Y_i. Then certainly $\Gamma_R(Q/R) = 1$, since we will have added no limits of elements of Y_1. Moreover, we will do the construction so that for each family $\{e_\sigma^\tau : \sigma < \tau < \omega_1\}$ as in (†), there is a family $\{c_\sigma : \sigma < \omega_1\}$ as in (††). There are only $(2^{\aleph_1})^{\aleph_1} = 2^{\aleph_1}$ possible families $\{e_\sigma^\tau : \sigma < \tau < \omega_1\}$ and the cofinality of $2^{\aleph_1} > \aleph_1$, so we can arrange our construction such that each such family is considered at some stage $i < 2^{\aleph_1}$ where $\{e_\sigma^\tau : \sigma < \tau < \omega_1\} \subseteq Y_i$.

Suppose now that Y_i has been constructed and that at this stage we are considering the family $\{e_\sigma^\tau : \sigma < \tau < \omega_1\} \subseteq Y_i$. Our plan is to let Y_{i+1} be the smallest valuation subring of $\widehat{R}$ which is closed under truncation and contains $Y_i \cup \{X^{(\alpha, i)} : \alpha < \omega_1\}$ and a family $\{c_\sigma : \sigma < \omega_1\}$ as in (††); we must choose $\{c_\sigma : \sigma < \omega_1\}$ so that $(\pounds)$ holds. We define the c_σ by induction. Let $c_0 = 1$; if c_σ has been defined, let

$$c_{\sigma+1} = e_\sigma^{\sigma+1} c_\sigma + X^{(\sigma, i)}.$$

If δ is a limit ordinal, let c_δ be the unique element of $\widehat{R}$ with support in $\bigoplus_{j<i} \bigoplus_{\alpha<\omega_1} \mathbb{Z}(\alpha, j)$ which represents the limit of $\langle e_\sigma^\delta c_\sigma : \sigma < \delta \rangle$ in $\widehat{R}/\bigcap_{\sigma<\delta} r_\sigma \widehat{R}$.

We sketch the argument that this construction works. First, by induction one can show that for all τ, $(\tau, i) \in p\text{-supp}(c_\sigma)$ if and only if $\tau < \sigma$. Secondly, the c_σ are algebraically independent over the quotient field of $\widehat{R}_i$. More generally, if $\sigma_1 < \cdots < \sigma_n < \delta \leq \sigma_{n+1}$, then $\{c_{\sigma_1}, \ldots, c_{\sigma_n}, c_{\sigma_{n+1}} \restriction \delta\}$ is algebraically independent over $\widehat{R}_i$. (This is proved by an argument on supports.) Now an arbitrary element of Y_{i+1} has the form

$$\frac{f(c_{\sigma_1}, \ldots, c_{\sigma_n}) \restriction (\delta, 0)}{g(c_{\sigma_1}, \ldots, c_{\sigma_n}) \restriction (\gamma, 0)}$$

where f and g have coefficients in Y_i. By an argument on supports, using the above facts, one shows that if this element belongs to $Y_{i+1} \cap \widehat{R}_i$, then it belongs to Y_i. □

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